Polyhedral Realizations of Crystal Bases for Modified Quantum Algebras of Type A

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Abstract

We describe the crystal bases of modified quantum algebras and its connected component containing "zero vector" by the polyhedral realization method for the type A_n and $A_1^{(1)}$. We also present the explicit form of the unique highest weight vector in the connected component.

1 Introduction

Let $U_q(\mathfrak{g}) := < e_i, f_i, q^h >_{i\in I}$ $(I = 1, 2, \cdots, n)$ be the quantum algebra associated with the symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ and $U_q^-(\mathfrak{g}) := < f_i >_{i\in I}$ (resp. $U_q^+(\mathfrak{g}) := < e_i >_{i\in I}$) be the subalgebra of $U_q(\mathfrak{g})$. Kashiwara showed that the subalgebras $U_q^{\pm}(\mathfrak{g})$ has a unique crystal base $(L(\mp\infty), B(\mp\infty))$ and arbitrary integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ has a unique crystal baase $(L(\lambda), B(\lambda))$ ([2]).

He also proved that a tensor product of crystal bases for modules is again a crystal base for tensor product of corresponding modules, which is one of the most beautiful and usefull properties in theory of crystal bases.

The term "crystal " implies a combinatorial notion abstracting the properties of crystal base without assuming existence of the corresponding modules. We shall see its examples in Sect.2.1. We can define tensor product structure on crystals in a similar manner to crystal bases. Indeed, some crystals (and their tensor products) are used to realize crystal base $B(\infty)$([8]) and $B(\lambda)$([7]) by the polyhedral realization method. Polyhedral realization of crystal bases is the method embedding crystal bases in some infinite-dimensional vector space and describing its image as a set of lattice points in certain convex polyhedron([7],[8]).

The modified quantum algebra $\tilde{U}_q(\mathfrak{g}) := \bigoplus_{\lambda\in P} U_q(\mathfrak{g})a_\lambda$ (resp. $U_q(\mathfrak{g})a_\lambda := U_q(\mathfrak{g})/\sum U_q(\mathfrak{g})(q^h - q^{\langle h_i,\lambda\rangle})$) is given by modifiying the Cartan part of $U_q(\mathfrak{g})$. Lusztig showed that it has a crystal base $(L(\tilde{U}_q(\mathfrak{g})), B(\tilde{U}_q(\mathfrak{g}))) = (\oplus_\lambda L(U_q(\mathfrak{g})a_\lambda), \oplus_\lambda B(U_q(\mathfrak{g})a_\lambda))$ ([6]) and Kashiwara describe its

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important properties ([5]). One of them is that the existence of the following isomorphism of crystals:

$$B(U_q(\mathfrak{g})a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty)$$

where T_λ is the crystal given in Sect.2.2. We have already the polyhedral realization of $B(\pm)$ and then we can get the polyhedral realization of $B(U_q(\mathfrak{g})a_\lambda)$ (Sect.4), which is, in general, not connected, on the contrary, including infinitely many components. So, next we try to describe some specific connected component $B_0(\lambda)$ containing $u_\infty \otimes t_\lambda \otimes u_{-\infty}$ and the explicit form of the unique highest weight vector in $B_0(\lambda)$ by the polyhedral realization method under certain assumptioin on the weight λ in the cases $\mathfrak{g} = A_n$ and $A_1^{(1)}$.

The organization of this paper is as follows: in Sect.2., we review the theory of crystal bases and crytals. We also prepare several ingredients to use in the subsequent sections, *e.g.,* crystals B_i, T_λ and explain polyhedral realization of $B(\pm\infty)$. In Sect.3., we define the modified quantum algebra $\tilde{U_q(\mathfrak{g})}$ and see its crystal base and in Sect.4., its polyhedral realization is given. In Sect.5.,we consider the modified quantum algebra of type A_n. We describe the highest weight vector in the connected component $B_0(\lambda)$ including $u_\infty \otimes t_\lambda \otimes u_{-\infty}$ and the polyhedal realization of $B_0(\lambda)$ under certain condition on λ. In the last section, we treat the affine $A_1^{(1)}$-case. In this case, we succeeded in presenting the explicit form of the connected component of $B_0(\lambda)$ and the unique highest weight vector in $B_0(\lambda)$ for a poisitive level weight λ.

2 Crystal Base and Crystals

2.1 Definition of Crystal Base and Crystal

In this section, we shall review crystal bases and crystals. We follow the notations and terminologies to [7][8].

We fix a finite index set I. Let $A = (a_{ij})_{i,j\in I}$ be a generalized symmetrizable Cartan matrix, $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{h_i\}_{i\in I})$ be the associated Cartan data and $\mathfrak{g}$ be the associated Kac-Moody Lie algebra where α_i(resp. h_i) is called an simple root (resp. simple coroot). Let P be a weight lattice, P^* be a dual lattice including $\{h_i\}_{\in I}$ and $Q := \bigoplus_{i\in I} \mathbf{Q}(q)\alpha_i$ be a root lattice. Let $U_q(\mathfrak{g}) := \langle q^h, e_i, f_i\rangle_{i\in I, h\in P^*}$ be the quantum algebra defined by the usual relations, which is an associative and Hopf algebra over the field $\mathbf{Q}(q)$. (We do not write down the Hopf algebra structure here.)

Now we give the definition of crystal base. Let $\mathcal{O}_{\rm int}$ be the category whose objects are $U_q(\mathfrak{g})$-module that it has a weight space decomposition and for any element u, there exists positive integer l such that $e_{i_1}\cdots e_{i_k}u = 0$ for any $k > l$. It is well-known that the category $\mathcal{O}_{\rm int}$ is a semisimple category and all simple ojbects are parametrized by the set of dominant integral weights P_+. Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\rm int}$. For any $u \in M_\lambda$ ($\lambda \in P$), we have the unique expression:

$$u = \sum_{n\geq 0} f_i^{(n)} u_n,$$

where $u_n \in \operatorname{Ker} e_i \cap M_{\lambda+n\alpha_i}$. By using this, we define the Kashiwara operators $\tilde{e}_i, \tilde{f}_i \in \operatorname{End}(M)$ $(i \in I)$,

$$\tilde{e}_i u := \sum_{n\geq 1} f_i^{(n-1)} u_n, \qquad \tilde{f}_i u := \sum_{n\geq 0} f_i^{(n+1)} u_n.$$

here note that we can define the Kashiwara operators $\tilde{e}_i, \tilde{f}_i \in \mathrm{End}(U_q^{\pm}(\mathfrak{g}))$ by the similar manner([2]). Let $A \subset \mathbf{Q}(q)$ be the subring of rational functions which are regular at $q=0$. Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\mathrm{int}}$ (resp. $U_q^{\pm}(\mathfrak{g})$).

Definition 2.1 ([2]). A pair (L, B) is a crystal base of M (resp. $U_q^{\pm}(\mathfrak{g})$), if it satisfies:

(i) L is free A-submodule of M (resp. $U_q^{\pm}(\mathfrak{g})$) and $M \cong \mathbf{Q}(q) \otimes_A L$ (resp. $U_q^{\pm}(\mathfrak{g}) \cong \mathbf{Q}(q) \otimes_A L$).

(ii) B is a basis of the $\mathbf{Q}$-vector space L/qL.

(iii) $L = \oplus_{\lambda \in P} L_\lambda$, $B = \sqcup_{\lambda \in P} B_\lambda$ where $L_\lambda := L \cap M_\lambda$, $B_\lambda := B \cap L_\lambda / qL$.

(iv) $\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$.

(v) $\tilde{e}_i B \subset B \sqcup \{0\}$ and $\tilde{f}_i B \subset B \sqcup \{0\}$ (resp. $\tilde{f}_i B \subset B$) ($\tilde{e}_i$ and $\tilde{f}_i$ acts on L/qL by (iv)).

(vi) For $u, v \in B$, $\tilde{f}_i u = v \Longleftrightarrow \tilde{e}_i v = u$.

The unit of subalgebra $U_q^{\mp}(\mathfrak{g})$ is denoted by $u_{\pm\infty}$. We set

$$
\begin{aligned}
&L(\infty) := \sum_{i_j \in I, l \geq 0} A \tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\infty, \\
&L(-\infty) := \sum_{i_j \in I, l \geq 0} A \tilde{e}_{i_l} \cdots \tilde{e}_{i_1} u_{-\infty}, \\
&B(\infty) := \{\tilde{f}_{i_l} \cdots \tilde{f}_{i_1} u_\infty \bmod qL(\infty) \,|\, i_j \in I, l \geq 0\}, \\
&B(-\infty) := \{\tilde{e}_{i_l} \cdots \tilde{e}_{i_1} u_{-\infty} \bmod qL(-\infty) \,|\, i_j \in I, l \geq 0\}.
\end{aligned}
$$

Theorem 2.2 ([2]). *A pair $(L(\pm\infty), B(\pm\infty))$ is a crystal base of $U_q^{\mp}(\mathfrak{g})$.*

Now we introduce the notion *crystal*, which is obtained by abstracting the combinatorial properties of crystal bases.

Definition 2.3. A *crystal* B is a set endowed with the following maps:

$$
\begin{aligned}
&wt : B \longrightarrow P, \\
&\varepsilon_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \longrightarrow \mathbf{Z} \sqcup \{-\infty\} \quad \text{for} \quad i \in I, \\
&\tilde{e}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\} \quad \text{for} \quad i \in I, \\
&\tilde{e}_i(0) = \tilde{f}_i(0) = 0.
\end{aligned}
$$

those maps satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

$$
\begin{aligned}
&\varphi_i(b) = \varepsilon_i(b) + \langle h_i, wt(b) \rangle, \\
&wt(\tilde{e}_i b) = wt(b) + \alpha_i \text{ if } \tilde{e}_i b \in B, \\
&wt(\tilde{f}_i b) = wt(b) - \alpha_i \text{ if } \tilde{f}_i b \in B, \\
&\tilde{e}_i b_2 = b_1 \Longleftrightarrow \tilde{f}_i b_1 = b_2 \ (b_1, b_2 \in B), \\
&\varepsilon_i(b) = -\infty \Longrightarrow \tilde{e}_i b = \tilde{f}_i b = 0.
\end{aligned}
$$

Indeed, if (L, B) is a crystal base, then B is a crystal.

Definition 2.4. (i) Let B_1 and B_2 be crystals. A *strict morphism* of crystals $\psi : B_1 \longrightarrow B_2$ is a map $\psi : B_1 \sqcup \{0\} \longrightarrow B_2 \sqcup \{0\}$ satisfying the following: (1) $\psi(0) = 0$. (2)If $b \in B_1$ and $\psi(b) \in B_2$, then

$$wt(\psi(b)) = wt(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b).$$

and the map ψ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$.

(ii) An injective strict morphism is called an *embedding of crystals*. We call B_1 is a subcrystal of B_2, if B_1 is a subset of B_2 and becomes a crystal itself by restricting the data on it from B_2.

The following examples will play an important role in the subsequent sections.

Example 2.5. *Let $T_\lambda := \{t_\lambda\}$ $(\lambda \in P)$ be the crystal consisting of one element t_λ defined by $wt(t_\lambda) = \lambda$, $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$, $\tilde{e}_i(t_\lambda) = \tilde{f}_i(t_\lambda) = 0$.*

Example 2.6. *For $i \in I$, the crystal $B_i := \{(x)_i \, : \, x \in \mathbf{Z}\}$ is defined by:*

$$\begin{array}{lll} wt((x)_i) = x\alpha_i, & \varepsilon_i((x)_i) = -x, & \varphi_i((x)_i) = x, \\ \varepsilon_j((x)_i) = -\infty, & \varphi_j((x)_i) = -\infty & \text{for} \quad j \neq i, \\ \tilde{e}_j(x)_i = \delta_{i,j}(x+1)_i, & \tilde{f}_j(x)_i = \delta_{i,j}(x-1)_i. & \end{array}$$

Note that as a set B_i is identidied with the set of integers $\mathbb{Z}$.

2.2 Polyhedral Realization of $B(\pm\infty)$

We review the polyhedral realization of the crystal $B(\pm\infty)$ following to [8].

We consider the following additive groups:

$$\mathbf{Z}^{+\infty} := \{(\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbf{Z} \text{ and } x_k = 0 \text{ for } k \gg 0\},$$

$$\mathbf{Z}^{-\infty} := \{(x_{-1}, x_{-2}, \cdots, x_{-k}, \cdots) : x_{-k} \in \mathbf{Z} \text{ and } x_{-k} = 0 \ for\ k \gg 0\}.$$

We will denote by $\mathbf{Z}^{+\infty}_{\geq 0} \subset \mathbf{Z}^{+\infty}$ (resp. $\mathbf{Z}^{-\infty}_{\leq 0} \subset \mathbf{Z}^{-\infty}$) the semigroup of non-negative (resp. non-positive) integer sequences. Take an infinite sequence of indices $\iota^+ = (\cdots, i_k, \cdots, i_2, i_1)$ (resp. $\iota^- = (i_{-1}, i_{-2}, \cdots, i_{-k}, \cdots)$) from I such that

$$i_k \neq i_{k+1} \text{ for any } k, \text{ and } \sharp\{k > 0 \text{ (resp. k} < 0) \ : i_k = i\} = \infty \text{ for any } i \in I. \tag{2.1}$$

We can associate to ι^+ (resp. ι^-) a crystal structure on $\mathbf{Z}^{+\infty}$ (resp. $\mathbf{Z}^{-\infty}$) (see [8]) and denote it by $\mathbf{Z}^{+\infty}_{\iota^+}$ (resp. $\mathbf{Z}^{-\infty}_{\iota^-}$). Let B_i be the crystal given in Example2.6. We obtain the following embeddings([3]):

$$\begin{array}{rcl} \Psi_i^+ : B(\infty) & \hookrightarrow & B(\infty) \otimes B_i \quad (u_\infty \mapsto u_\infty \otimes (0)_i), \\ \Psi_i^- : B(-\infty) & \hookrightarrow & B_i \otimes B(-\infty) \quad (u_{-\infty} \mapsto (0)_i \otimes u_{-\infty}). \end{array}$$

Iterating Ψ_i^+ (resp. Ψ_i^-) according to ι^+ (resp. ι^-), we get the *Kashiwara embedding*([3]);

$$\Psi_{\iota^+} : B(\infty) \hookrightarrow \mathbf{Z}_{\geq 0}^{+\infty} \subset \mathbf{Z}_{\iota^+}^{+\infty} \quad (u_\infty \mapsto (\cdots, 0, \cdots, 0, 0, 0)), \tag{2.2}$$

$$\Psi_{\iota^-} : B(-\infty) \hookrightarrow \mathbf{Z}_{\leq 0}^{-\infty} \subset \mathbf{Z}_{\iota^-}^{-\infty} \quad (u_{-\infty} \mapsto (0, 0, 0, \cdots, 0, \cdots)). \tag{2.3}$$

We consider the following infinite dimensional vector spaces and their dual spaces:

$$\begin{aligned}
\mathbf{Q}^{+\infty} &:= \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}^{+\infty} = \{\vec{x} = (\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbf{Q} \text{ and } x_k = 0 \text{ for } k \gg 0\},\\
\mathbf{Q}^{-\infty} &:= \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}^{-\infty} = \{\vec{x} = (x_{-1}, x_{-2}, \cdots, x_{-k}, \cdots) : x_{-k} \in \mathbf{Q} \text{ and } x_{-k} = 0 \text{ for } k \gg 0\},\\
(\mathbf{Q}^{\pm\infty})^* &:= \mathrm{Hom}(\mathbf{Q}^{\pm\infty}, \mathbf{Q}).
\end{aligned}$$

We will write a linear form $\varphi \in (\mathbf{Q}^{+\infty})^*$ as $\varphi(\vec{x}) = \sum_{k \geq 1} \varphi_k x_k$ $(\varphi_j \in \mathbf{Q})$. Similarly, we write $\varphi \in (\mathbf{Q}^{-\infty})^*$ as $\varphi(\vec{x}) = \sum_{k \leq -1} \varphi_k x_k$ $(\varphi_j \in \mathbf{Q})$.

For the sequence $\iota^+ = (i_k)_{k \geq 1}$ (resp. $\iota^- = (i_k)_{k \leq -1}$) and $k \geq 1$ (resp. $k \leq -1$), we set

$$k^{(+)} := \min\{l : l > k > 0 \text{ (resp. } 0 > l > k) \text{ and } i_k = i_l\},$$

if it exists, and

$$k^{(-)} := \max\{l : 0 < l < k \text{ (resp. } \mathrm{l} < \mathrm{k} < 0) \text{ and } i_k = i_l\},$$

if it exists, otherwise $k^{(+)} = k^{(-)} = 0$.

We define a linear form β_k $(k \geq 0)$ on $\mathbf{Q}^{+\infty}$ by

$$\beta_k(\vec{x}) := \begin{cases} x_k + \sum_{k<j<k^{(+)}} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^{(+)}} & (k \geq 1), \\ 0 & (k = 0). \end{cases} \tag{2.4}$$

We also define a linear form β_k $(k \leq 0)$ on $\mathbf{Q}^{-\infty}$ by

$$\beta_k(\vec{y}) := \begin{cases} y_{k^{(-)}} + \sum_{k^{(-)}<j<k} \langle h_{i_k}, \alpha_{i_j} \rangle y_j + y_k & (k \leq -1), \\ 0 & (k = 0). \end{cases} \tag{2.5}$$

By using these linear forms, let us define a piecewise-linear operator $S_k = S_{k,\iota}$ on $(\mathbf{Q}^{\pm\infty})^*$ as follows:

$$S_k(\varphi) := \begin{cases} \varphi - \varphi_k \beta_k & if\ \varphi_k > 0, \\ \varphi - \varphi_k \beta_{k^{(-)}} & if\ \varphi_k \leq 0, \end{cases} \tag{2.6}$$

for $\varphi(\vec{x}) = \sum \varphi_k x_k \in (\mathbf{Q}^{\pm\infty})^*$. Here we set

$$\begin{aligned}
\Xi_{\iota^\pm} &:= \{S_{\pm j_l} \cdots S_{\pm j_2} S_{\pm j_1}(\pm x_{j_0}) \,|\, l \geq 0, j_0, j_1, \cdots, j_l \geq 1\},\\
\Sigma_{\iota^\pm} &:= \{\vec{x} \in \mathbf{Z}^{\pm\infty} \subset \mathbf{Q}^{\pm\infty} \,|\, \varphi(\vec{x}) \geq 0 \text{ for any } \varphi \in \Xi_{\iota^\pm}\},
\end{aligned}$$

We impose on ι^+ and ι^- the following assumptions (P),(N):

(P) for ι^+, if a positive k satisfies $k^{(-)} = 0$ then $\varphi_k \geq 0$ for any $\varphi(\vec{x}) = \sum_k \varphi_k x_k \in \Xi_{\iota^+}$,

(N) for ι^-, if a negative k satisfies $k^{(+)} = 0$ then $\varphi_k \leq 0$ for any $\varphi(\vec{x}) = \sum_k \varphi_k x_k \in \Xi_{\iota^-}$.

Theorem 2.7 ([8]). *Let $\iota^\pm$ be the indices of sequences which are satisfied (2.1) and the assumptions (P),(N). Suppose $\Psi_{\iota^+} : B(\infty) \hookrightarrow \mathbf{Z}_{\iota^+}^{\infty}$ and $\Psi_{\iota^-} : B(-\infty) \hookrightarrow \mathbf{Z}_{\iota^-}^{-\infty}$ are the Kashiwara embeddings. Then, we have* $\mathrm{Im}(\Psi_{\iota^+})(\cong B(\infty)) = \Sigma_{\iota^+}$, $\mathrm{Im}(\Psi_{\iota^-})(\cong B(-\infty)) = \Sigma_{\iota^-}$.

3 Modified Quantum Algebras and its Crystal Base

We define the left $U_q(\mathfrak{g})$-module $U_q(\mathfrak{g})a_\lambda$ ([4]) by the relation: $q^h a_\lambda = q^{\langle h,\lambda\rangle}a_\lambda$. Then $\tilde{U}_q(\mathfrak{g}) = \oplus_{\lambda\in P}U_q(\mathfrak{g})a_\lambda$ has an algebra structure by

(i) $a_\lambda P = P a_{\lambda-\xi}$ for $\xi \in Q$ and $P \in U_q(\mathfrak{g})_\xi$
$(\, U_q(\mathfrak{g})_\xi := \{P \in U_q(\mathfrak{g});\, q^h P q^{-h} = q^{\langle h,\xi\rangle}P$ for any $h \in P^*\}\,)$.

(ii) $a_\lambda a_\mu = \delta_{\lambda,\mu}a_\lambda$,

and we call this algebra *modified quantum algebra*.

Let M be a $U_q(\mathfrak{g})$-module with the weight space decomposition $M = \oplus_{\lambda\in P}M_\lambda$. Then a_λ is a projection $a_\lambda : M \longrightarrow M_\lambda$.

In [6], it is revealed that modified quantum algebra $\tilde{U}_q(\mathfrak{g})$ has a crystal structure and in [4] its crystal base is described as follows:

Theorem 3.1 ([4]).

$$\begin{aligned} B(U_q(\mathfrak{g})a_\lambda) &= B(\infty)\otimes T_\lambda \otimes B(-\infty), \\ B(\tilde{U}_q(\mathfrak{g})) &= \bigoplus_{\lambda\in P} B(\infty)\otimes T_\lambda \otimes B(-\infty). \end{aligned}$$

4 Polyhedral Realization of $B(U_q(\mathfrak{g})a_\lambda)$

4.1 Crystal structure of $\mathbf{Z}^\infty[\lambda]$

Let $\mathbf{Z}^{+\infty}_{\iota^+}$ and $\mathbf{Z}^{-\infty}_{\iota^-}$ be as in Sect.2.2, We take the indices sequence $\iota := (\iota^+, t_\lambda, \iota^-) = (\cdots, i_2, i_1, t_\lambda, i_{-1}, i_{-2}, \cdots)$ and weight $\lambda \in P$. We set $\mathbf{Z}^\infty_\iota[\lambda] := \mathbf{Z}^\infty_{\iota^+}\otimes T_\lambda \otimes \mathbf{Z}^{-\infty}_{\iota^-}$. The crystal structure on $\mathbf{Z}^\infty_\iota[\lambda]$ associated with ι and λ is defined as follows: We identify $\mathbf{Z}^{+\infty}\otimes T_\lambda\otimes\mathbf{Z}^{-\infty}$ with $\mathbf{Z}^\infty$. Therefore, $\mathbf{Z}^\infty[\lambda]$ is regarded as a sublattice of $\mathbf{Q}^\infty = \mathbf{Q}\otimes_{\mathbf{Z}}\mathbf{Z}^\infty$. Thus, we can deonte $\vec{x}\in\mathbf{Z}^\infty_\iota[\lambda]$ by $\vec{x} = (\cdots, x_2, x_1, t_\lambda, x_{-1}, x_{-2}, \cdots)$. For $\vec{x}\in\mathbf{Q}^\infty$, we define a linear function $\sigma_k(\vec{x})$ $(k\in\mathbf{Z})$ by:

$$\sigma_k(\vec{x}) := \begin{cases} x_k + \sum_{j>k}\langle h_{i_k}, \alpha_{i_j}\rangle x_j & (k\geq 1), \\ -\langle h_{i_k},\lambda\rangle + x_k + \sum_{j>k}\langle h_{i_k},\alpha_{i_j}\rangle x_j & (k\leq -1), \\ -\infty & (k=0). \end{cases} \tag{4.1}$$

Since $x_j = 0$ for $j \gg 0$, σ_k is well-defined. Let $\sigma^{(i)}(\vec{x}) := \max_{k:i_k=i}\sigma_k(\vec{x})$ and

$$M^{(i)} = M^{(i)}(\vec{x}) := \{k : i_k = i, \sigma_k(\vec{x}) = \sigma^{(i)}(\vec{x})\}. \tag{4.2}$$

Note that $\sigma^{(i)}(\vec{x}) \geq 0$, and that $M^{(i)} = M^{(i)}(\vec{x})$ is a finite set if and only if $\sigma^{(i)}(\vec{x}) > 0$. Now, we define the map $\tilde{e}_i : \mathbf{Z}^\infty[\lambda]\sqcup\{0\} \longrightarrow \mathbf{Z}^\infty[\lambda]\sqcup\{0\}$, $\tilde{f}_i : \mathbf{Z}^\infty[\lambda]\sqcup\{0\} \longrightarrow \mathbf{Z}^\infty[\lambda]\sqcup\{0\}$, by $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$ and

$$(\tilde{f}_i(\vec{x}))_k = x_k + \delta_{k,\min M^{(i)}} \text{ if } M^{(i)} \text{ exists; otherwise } \tilde{f}_i(\vec{x}) = 0, \tag{4.3}$$

$$(\tilde{e}_i(\vec{x}))_k = x_k - \delta_{k,\max M^{(i)}} \text{ if } M^{(i)} \text{ exists; otherwise } \tilde{e}_i(\vec{x}) = 0. \tag{4.4}$$

where $\delta_{i,j}$ is Kronecker's delta. We also define the weight function and the function ε_i and φ_i on $\mathbf{Z}^\infty[\lambda]$ as follows:

$$\begin{array}{l} wt(\vec{x}) := \lambda - \sum_{j=-\infty}^{\infty} x_j \alpha_{i_j},\ \ \varepsilon_i(\vec{x}) := \sigma^{(i)}(\vec{x}), \\ \varphi_i(\vec{x}) := \langle h_i, wt(\vec{x})\rangle + \varepsilon_i(\vec{x}). \end{array} \tag{4.5}$$

We denote this crystal by $\mathbf{Z}_\iota^\infty[\lambda]$.

Since there exist the embedings of crystals: $B(\pm) \hookrightarrow \mathbf{Z}_{\iota^\pm}^{\pm\infty}$, we obtain

Theorem 4.1.

$$\begin{array}{rcl} \Psi_\iota^{(\lambda)} : B(\infty) \otimes T_\lambda \otimes B(-\infty) & \hookrightarrow & \mathbf{Z}_{\iota^+}^{+\infty} \otimes T_\lambda \otimes \mathbf{Z}_{\iota^-}^{-\infty} (= \mathbf{Z}_\iota^\infty[\lambda]) \\ u_\infty \otimes t_\lambda \otimes u_{-\infty} & \mapsto & (\cdots, 0, 0, t_\lambda, 0, 0, \cdots) \end{array}$$

is the unique stirict embedding which is associated with $\iota := (\cdots, i_2, i_1, t_\lambda, i_{-1}, i_{-2}, \cdots)$.

4.2 The image of $\Psi_\iota^{(\lambda)}$

Fix a sequence of indices ι as above. We define a linear function $\bar{\beta}_k(\vec{x})$ as follows:

$$\bar{\beta}_k(\vec{x}) \ \ = \ \ \sigma_k(\vec{x}) - \sigma_{k^{(+)}}(\vec{x}) \tag{4.6}$$

where σ_k is defined by (4.1). Since $\langle h_i, \alpha_i\rangle = 2$ for any $i \in I$, we have

$$\bar{\beta}_k(\vec{x}) \ \ = \ \ \begin{cases} x_k + \sum_{k<j<k^{(+)}} \langle h_{i_k}, \alpha_{i_j}\rangle x_j + x_{k^{(+)}} & (k \geq 1 \ or \ k^{(+)} \leq -1), \\ -\langle h_{i_k}, \lambda\rangle + x_k + \sum_{k<j<k^{(+)}} \langle h_{i_k}, \alpha_{i_j}\rangle x_j + x_{k^{(+)}} & (k \leq -1 \ and \ k^{(+)} > 0). \end{cases}$$

Using this notation, we define an operator $\bar{S}_k = \bar{S}_{k,\iota}$ for a linear function $\varphi(\vec{x}) = c + \sum_{-\infty}^{\infty} \varphi_k x_k$ $(c, \varphi_k \in \mathbf{Q})$ as follows:

$$\bar{S}_k\,(\varphi) := \begin{cases} \varphi - \varphi_k \bar{\beta}_k & if\ \ \varphi_k > 0, \\ \varphi - \varphi_k \bar{\beta}_{k^{(-)}} & if\ \ \varphi_k \leq 0. \end{cases} \tag{4.7}$$

An easy check shows $(\bar{S}_k)^2 = \bar{S}_k$. For a sequence ι and an integral weight λ, we denote by $\Xi_\iota[\lambda]$ the subset of linear forms which are obtained from the coordinate forms x_j, x_{-j} $(j \geq 1)$ by applying transformations $\bar{S}_k$. In other words, we set

$$\begin{array}{ll} \Xi_\iota^+[\lambda] & := \{\bar{S}_{j_l} \cdots \bar{S}_{j_1}(x_{j_0}) \,:\, l \geq 0,\ j_0, \cdots, j_l > 0\} \\ \Xi_\iota^-[\lambda] & := \{\bar{S}_{-j_k} \cdots \bar{S}_{-j_1}(-x_{-j_0}) \,:\, k \geq 0,\ j_0, \cdots, j_k > 0\}, \\ \Xi_\iota[\lambda] & := \Xi_\iota^+[\lambda] \cup \Xi_\iota^-[\lambda]. \end{array} \tag{4.8}$$

Now we set

$$\Sigma_\iota[\lambda] := \{\vec{x} \in \mathbf{Z}_\iota^\infty[\lambda] (\subset \mathbf{Q}^\infty) \,:\, \varphi(\vec{x}) \geq 0 \text{ for any } \varphi \in \Xi_\iota[\lambda]\}. \tag{4.9}$$

By Theorem 2.7, we have

Theorem 4.2. *Suppose that $\iota^{\pm}$ satisfys the assumption (P),(N),and (2.1).Let $\Psi_\iota^{(\lambda)} : B(\infty)\otimes T_\lambda \otimes B(-\infty) \hookrightarrow \mathbf{Z}_\iota^\infty[\lambda]$ be the embedding of (4.1). Then $\mathrm{Im}(\Psi_\iota^{(\lambda)})(\cong B(\infty)\otimes T_\lambda\otimes B(-\infty))$ is equal to $\Sigma_\iota[\lambda]$.*

Remark. Under the assumptions (P) and (N), both $\Xi_\iota^+[\lambda]$ and $\Xi_\iota^-[\lambda]$ are closed by the actions of $\bar{S}_k$'s, since S_k $(k<0)$ (resp. S_k $(k>0)$) acts identically on $\Xi_\iota^+[\lambda]$ (resp. $\Xi_\iota^-[\lambda]$).

The following lemma will be used lalter.

Lemma 4.3. *Let Ξ be a set of linear functions on $\mathbf{Q}^\infty$. Suppose that Ξ is closed by actions of all $\bar{S}_k$'s, then the set*

$$\Sigma = \{\vec{x}\in \mathbf{Z}_\iota^\infty[\lambda] | \varphi(\vec{x})\geq 0 \text{ for any } \varphi\in\Xi_\iota\}$$

is a sub-crystal of $\mathbf{Z}_\iota^\infty[\lambda]$.

Proof. It suffices to show that Σ is closed under the actions of $\tilde{f}_i$ and $\tilde{e}_i$. For $\vec{x}\in\Sigma$, suppose $\tilde{f}_i\vec{x} = (\cdots, x_k+1,\cdots)$. For any $\varphi = c+\sum\varphi_j x_j \in \Xi$ $(c,\varphi_j\in\mathbf{Q})$, we need to show that

$$\varphi(\tilde{f}_i\vec{x})\geq 0. \tag{4.10}$$

Since $\varphi(\tilde{f}_i(\vec{x})) = \varphi(\vec{x})+\varphi_k \geq \varphi_k$, it is enough to consider the case when $\varphi_k<0$. By definition of $\tilde{f}_i\vec{x}$, we have $\sigma_{k(-)}<\sigma_k$. This shows that

$$\begin{aligned} \sigma_{k(-)}<\sigma_k &\iff \bar{\beta}_{k(-)} = \sigma_{k(-)}-\sigma_k < 0 \\ &\implies \bar{\beta}_{k(-)}\leq -1. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \varphi(\tilde{f}_i\vec{x}) &= \varphi(\vec{x})+\varphi_k \\ &\geq \varphi(\vec{x})-\varphi_k\bar{\beta}_{k(-)} \\ &= (\bar{S}_k\varphi)(\vec{x}) \ \geq 0. \end{aligned}$$

Suppose that $\tilde{e}_i\vec{x} = (\cdots,x_k-1,\cdots)$. We need to show that

$$\varphi(\tilde{e}_i\vec{x})\geq 0. \tag{4.11}$$

Since $\varphi(\tilde{e}_i(\vec{x})) = \varphi(\vec{x})-\varphi_k\geq -\varphi_k$, it is enough to consider the case when $\varphi_k>0$. By definition of $\tilde{e}_i\vec{x}$, we have $\sigma_k>\sigma_{k(+)}$. This shows that

$$\begin{aligned} \sigma_k>\sigma_{k(+)} &\iff \bar{\beta}_k = \sigma_k-\sigma_{k(+)}>0 \\ &\implies \bar{\beta}_{k(-)}\geq 1. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \varphi(\tilde{e}_i\vec{x}) &= \varphi(\vec{x})-\varphi_k \\ &\geq \varphi(\vec{x})-\varphi_k\bar{\beta}_k \\ &= (\bar{S}_k\varphi)(\vec{x}) \ \geq 0. \end{aligned}$$

□

5 Polyhedral Realization of $B(U_q(\mathfrak{g})a_\lambda)$ of Type A_n

In this section, we shall describe the crystal structure of the component including t_λ in $B(\infty)\otimes T_\lambda\otimes B(-\infty)$ for the case of type A_n.

It will be convenient for us to change the indexing set for $\mathbf{Z}^\infty$ from $\mathbf{Z}_{\geq 1}$ to $\mathbf{Z}_{\geq 1}\times[1,n]$. We will do this with the help of the bijection $\mathbf{Z}_{\geq 1}\times[1,n]\to\mathbf{Z}_{\geq 1}$ given by $((j;i)\mapsto(j-1)n+i)$. Thus, we will write an element $\vec{x}\in\mathbf{Z}^{+\infty}$ as doubly-indexed family $(x_{j;i})_{j\geq 1,i\in[1,n]}$ of nonnegative integers. Simillarly, using that $\mathbf{Z}_{\geq 1}\times[1,n]\to\mathbf{Z}_{\leq -1}$ $((j;i)\mapsto -jn+i-1)$ is bijective, we will write an element $\vec{x}\in\mathbf{Z}^{-\infty}$ as doubly-indexed family $(x_{-j;i})_{j\geq 1,i\in[1,n]}$ of nonpositive integers. Therefore, we can write that $\vec{x}\in\mathbf{Z}^\infty$ as $(\cdots,x_{1;2},x_{1;1},t_\lambda,x_{-1;n},x_{-1;n-1},\cdots)$. We will adopt the convention that $x_{j;0}=x_{j;n+1}=0$ unless $i\in[1,n]$

To state the main theorem, we prepare several things. For $x\in\mathbb{R}$, set $(x)_+:=max(0,x)$. Let $\lambda=\sum_{1\leq i\leq n}\lambda_i\Lambda_i$ be an integral weight satisfying $\lambda_1,\cdots,\lambda_{i_0}>0$ and $\lambda_{i_0+1},\cdots,\lambda_n\leq 0$ for some i_0 and for $(j;i)\in\mathbf{Z}_{\geq 1}\times[1,n]$ set

$$C_{-j;i}=\begin{cases}(-\lambda_{-j+i+1}+(-\lambda_{-j+i+2}+(\cdots+(-\lambda_{-j+n+1})_+)_+\cdots)_+)_+ & \text{if } 1\leq j\leq i\leq n,\\ 0 & \text{otherwise.}\end{cases}$$

We will use the following lemma frequently:

Lemma 5.1. *For real numbers $r_1,\cdots,r_n$, we have,*

$$r_1+(r_2+(r_3+\cdots+(r_{n-1}+(r_n)_+)_+)_+)_+=\max(r_1,r_1+r_2,\cdots,r_1+r_2+\cdots+r_n)$$

Proof.
We can easily show from the fact : $r_1+(r_2)_+=\max(r_1,r_1+r_2)$ and iterating this. □

By the above lemma, we can write

$$C_{-j;i}=\max(0,-\lambda_{-j+i+1},-\lambda_{-j+i+1}-\lambda_{-j+i+2},\cdots,-\lambda_{-j+i+1}-\lambda_{-j+i+2}-\cdots-\lambda_{-j+n+1}).$$

Theorem 5.2. *Let $\iota=(\cdots,2,1,n,\cdots,2,1,t_\lambda,n,n-1,\cdots,1,n,n-1,\cdots)$ be an infinite sequence and λ and $C_{-j;i}$ be as above. We define*

$$\begin{aligned}\Xi'_\iota[\lambda] &:= \{\bar{S}_{-j_k}\cdots\bar{S}_{-j_1}(x_{-j;i}+C_{-j;i})\,:\,k\geq 0,\,i\in I,j\geq 1,\;j_1,\cdots,j_k\geq 1\},\\ \Sigma'_\iota[\lambda] &:= \{\vec{x}\in\mathbf{Z}^\infty_\iota[\lambda](\subset\mathbf{Q}^\infty)\,:\,\varphi(\vec{x})\geq 0\text{ for any }\varphi\in\Xi'_\iota[\lambda]\}\end{aligned}$$

and denote the connected component of $\mathrm{Im}\,(\Psi^{(\lambda)}_\iota)$ containing $\vec{0}:=(\cdots,0,0,t_\lambda,0,0,\cdots)$ by $B_0(\lambda)$. Then we have

(i) $B_0(\lambda)=\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$.

(ii) Let v_λ be the unique highest weight vector in $B_0(\lambda)$. Then we have

$$v_\lambda=(\cdots,0,0,t_\lambda,-C_{-1;n},-C_{-1;n-1},\cdots,-C_{-j;i},\cdots),$$

Proof. Since $\Xi'_\iota[\lambda]$ is closed under the actions of $\bar{S}_k$'s, by Lemma 4.3 $\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$ has a crystal structure unless it is empty. We will show that $\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$ contains $\vec{0}$, which implies that $\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$ is non-empty, and has the unique highest weight vector. First, we will show that

$$x_0=(\cdots,0,0,t_\lambda,-C_{-1;n},-C_{-1;n-1},\cdots,-C_{-j;i},\cdots) \tag{5.1}$$

is a highest weight vector in $\mathbf{Z}^\infty_\iota[\lambda]$. We set $y_{-j;i}:=-\lambda_{-j+i+1}+(-\lambda_{-j+i+2}+(\cdots+(-\lambda_{-j+n+1})_+)_+)_+$. Thus, we have $C_{-j;i}=-(y_{-j;i})_+$. Due to (4.4), it suffices to show

$$\sigma_{-j:i}(x_0)\le 0 \text{ for } j\ge 1,\ i\in I.$$

(In the case $j<0$, trivially $\sigma_{-j:i}(x_0)=0$.) We consider the following four cases:
(I) $j=1$. (II) $i=n$. (III) $1\le i<j\le n$. (IV) $1<j\le i<n$.

(I) The case $j=1$.
We will show that $\sigma_{-1,n}(x_0),\sigma_{-1;n-1}(x_0),\cdots,\sigma_{-1;1}(x_0)\le 0$. Note the following simple fact:

$$-(-a)_+-a\le 0,\quad \text{for any } a\in\mathbb{R}. \tag{5.2}$$

We can write $y_{-1;i+1}:=-\lambda_{i+1}+(-\lambda_{i+2}+(\cdots+(-\lambda_n)_+)_+\cdots)_+$. By the definition of $\sigma_{-j:i}$, we have $\sigma_{-1;i}(x_0)=-(-\lambda_i+(y_{-1;i+1})_+)_++(y_{-1;i+1})_+-\lambda_i$. By (5.2), we obtain $\sigma_{-1;i}(x_0)\le 0$. This shows $\sigma_{-1;n}(x_0),\sigma_{-1;n-1}(x_0),\cdots,\sigma_{-1;1}(x_0)\le 0$.

(II) The case $i=n$.
We shall show $\sigma_{-j;n}(x_0)\le 0$ $(1\le\forall j\le n)$ by the induction on j. If $j=1$, it is true by (I). Suppose $j>1$. We can write $\sigma_{-j;n}(x_0)=-(-\lambda_{-j+n+1})_++(-\lambda_{-j+n+1}+(-\lambda_{-j+n+2})_+)_+-(-\lambda_{-j+n+2})_++\sigma_{-j+1;n}(x_0)$. Now, set $A:=\sigma_{-j;n}(x_0)-\sigma_{-j+1;n}(x_0)$. Since $\sigma_{-j+1;n}(x_0)\le 0$ by the induction hypothesis, it is sufficient to show $A\le 0$.

If $\lambda_{-j+n+1}\,,\,\lambda_{-j+n+2}\ge 0$ $\lambda_{-j+n+1}\,,\,\lambda_{-j+n+2}\le 0$, then obviously $A=0$. If $\lambda_{-j+n+1}\ge 0$ and $\lambda_{-j+n+2}\le 0$, we can write $A=(-\lambda_{-j+n+1}-\lambda_{-j+n+2})_++\lambda_{-j+n+2}$. In this case, if $-\lambda_{-j+n+1}-\lambda_{-j+n+2}\le 0$, then $A=\lambda_{-j+n+2}\le 0$. If $-\lambda_{-j+n+1}-\lambda_{-j+n+2}\ge 0$, then $A=-\lambda_{-j+n+1}\le 0$. If $\lambda_{-j+n+1}\le 0$ and $\lambda_{-j+n+2}\ge 0$, obviously $A=0$.

(III) The case $1\le i<j\le n$.
By the definition, $C_{-j;i}=0$ for $i<j$. We can write

$$\sigma_{-j;i}(x_0)=C_{-i-1:i+1}-C_{-i:i}+\sigma_{-i;i}(x_0).$$

By Lemma 5.1, we have

$$\begin{aligned} C_{-i:i}&=\max(0,-\lambda_1,-\lambda_1-\lambda_2,\cdots,-\lambda_1-\lambda_2-\cdots-\lambda_{-i+n+1}),\\ C_{-i-1:i+1}&=\max(0,-\lambda_1,-\lambda_1-\lambda_2,\cdots,-\lambda_1-\lambda_2-\cdots-\lambda_{-i+n}).\end{aligned}$$

Therefore, we obtain $C_{-i:i}\ge C_{-i-1:i+1}$. This shows $\sigma_{-j;i}(x_0)\le 0$.

(IV) The case $1<j\le i<n$.
We will show $\sigma_{-j;i}(x_0)\le 0$ by the induction on $1<j\le i<n$. We can write:

$$\sigma_{-j;i}(x_0)\ =\ -C_{-j;i}+C_{-j;i+1}+C_{-j+1;i-1}-C_{-j+1;i}+\sigma_{-j+1;i}(x_0)$$

where

$$\begin{aligned}
C_{-j;i} &= (-\lambda_{-j+i+1} + (-\lambda_{-j+i+2} + \cdots + (-\lambda_{-j+n+1})_+ \cdots)_+)_+, \\
C_{-j;i+1} &= (-\lambda_{-j+i+2} + (-\lambda_{-j+i+2} + \cdots + (-\lambda_{-j+n+1})_+ \cdots)_+)_+, \\
C_{-j+1;i-1} &= (-\lambda_{-j+i+1} + (-\lambda_{-j+i+2} + \cdots + (-\lambda_{-j+n+1} + (-\lambda_{-j+n+2})_+)_+ \cdots)_+)_+, \\
C_{-j+1;i} &= (-\lambda_{-j+i+2} + (-\lambda_{-j+i+3} + \cdots + (-\lambda_{-j+n+1} + (-\lambda_{-j+n+2})_+)_+ \cdots)_+)_+.
\end{aligned}$$

Set $B := \sigma_{-j;i}(x_0) - \sigma_{-j+1;i}(x_0)$. It is sufficient to show $B \leq 0$, since $\sigma_{-j+1;i}(x_0) \leq 0$ by the induction hypothesis. By Lemma 5.1, we have

$$\begin{aligned}
C_{-j;i+1} &= \max(0, -\lambda_{-j+i+2}, -\lambda_{-j+i+2} - \lambda_{-j+i+3}, \cdots, -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+1}), \\
C_{-j+1;i} &= \max(0, -\lambda_{-j+i+2}, -\lambda_{-j+i+2} - \lambda_{-j+i+3}, \cdots, -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+2}),
\end{aligned}$$

and then

$$C_{-j;i+1} \leq C_{-j+1;i}. \tag{5.3}$$

Since

$$B = -(-\lambda_{-j+i+1} + C_{-j;i+1})_+ + C_{-j;i+1} + (-\lambda_{-j+i+1} + C_{-j+1;i})_+ - C_{-j+1;i},$$

if $C_{-j;i+1} = C_{-j+1;i}$, then $B = 0$. We consider the case $C_{-j;i+1} \neq C_{-j+1;i}$. In this case, we have $C_{-j;i+1} < C_{-j+1;i}$ by (5.3). We can write

$$\begin{aligned}
C_{-j;i+1} &= 0 \ \text{ or } \ -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+k+1} > 0 \ \text{ for some } k \, (i \leq k \leq n), \\
C_{-j+1;i} &= -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+2} > 0, \\
C_{-j;i+1} - C_{-j+1;i} &= \lambda_{-j+k+2} + \lambda_{-j+k+3} + \cdots + \lambda_{-j+n+2} < 0. \qquad (5.4)
\end{aligned}$$

If $C_{-j;i+1} = 0$, then

$$\begin{aligned}
B &= -(-\lambda_{-j+i+1})_+ + (-\lambda_{-j+i+1} + C_{-j+1;i})_+ - C_{-j+1;i} \\
&= -(-\lambda_{-j+i+1})_+ + (-\lambda_{-j+i+1} - \lambda_{-j+i+2} - \cdots - \lambda_{-j+n+2})_+ \\
&\quad + \lambda_{-j+i+2} + \lambda_{-j+i+3} + \cdots + \lambda_{-j+n+2}.
\end{aligned}$$

In this case, if $-\lambda_{-j+i+1} - \lambda_{-j+i+2} - \cdots - \lambda_{-j+n+2} \leq 0$, then $-\lambda_{-j+i+1} \leq 0$ by (5.4). Therefore, we have $B < 0$. If $-\lambda_{-j+i+1} - \lambda_{-j+i+2} - \cdots - \lambda_{-j+n+2} > 0$, then $B = -(-\lambda_{-j+i+1})_+ - \lambda_{-j+i+1} \leq 0$ by (5.2). Therefore, in the case $C_{-j;i+1} = 0$, We get $B \leq 0$.

We consider the case $C_{-j;i+1} \neq 0$. In this case, we have

$$\begin{aligned}
B &= -(-\lambda_{-j+i+1} - \lambda_{-j+i+2} - \cdots - \lambda_{-j+k+1})_+ \\
&\quad + (-\lambda_{-j+i+1} - \lambda_{-j+i+2} - \cdots - \lambda_{-j+k+1} - \lambda_{-j+k+2} - \cdots - \lambda_{-j+n+2})_+ \\
&\quad + \lambda_{-j+k+2} + \lambda_{-j+k+3} + \cdots + \lambda_{-j+n+2}.
\end{aligned}$$

If $-\lambda_{-j+i+1} - \lambda_{-j+i+2} - \cdots - \lambda_{-j+k+1} \geq 0$, by (5.4), we have

$$\begin{aligned}
B &= \lambda_{-j+i+1} + \lambda_{-j+i+2} + \cdots + \lambda_{-j+k+1} \\
&\quad - \lambda_{-j+i+1} - \lambda_{-j+i+2} - \cdots - \lambda_{-j+k+1} - \lambda_{-j+k+2} - \cdots - \lambda_{-j+n+2} \\
&\quad + \lambda_{-j+k+2} + \lambda_{-j+k+3} + \cdots + \lambda_{-j+n+2} \\
&= 0.
\end{aligned}$$

On the other hand, if $-\lambda_{-j+i+1}-\lambda_{-j+i+2}-\cdots-\lambda_{-j+k+1}<0$, by (5.4), we have

$$\begin{aligned} B &= (-\lambda_{-j+i+1}-\lambda_{-j+i+2}-\cdots-\lambda_{-j+k+1}-\lambda_{-j+k+2}-\cdots-\lambda_{-j+n+2})_+ \\ &\quad +\lambda_{-j+k+2}+\lambda_{-j+k+3}+\cdots+\lambda_{-j+n+2} \\ &< 0. \end{aligned}$$

This shows $B\le 0$.

Therefore, we conclude $\sigma_{-j;i}(x_0)=B+\sigma_{-j+1;i}(x_0)\le 0$ for any j,i $(1\le i,j\le n)$ by the induction hypothesis. Now, we have shown that $x_0=(\cdots,0,0,t_\lambda,-C_{-1;n},-C_{-1;n-1},\cdots,-C_{-j;i},\cdots)$ is one of the highest weight vecters in $\mathbf{Z}^\infty_\iota[\lambda]$.

Next, we will show that $\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$ contains $\vec{0}$.

We know that $\vec{0}\in\Sigma_\iota[\lambda]$, since all the constant terms of all elements in $\Xi_\iota[\lambda]$ are 0. Therefore, we will show $\vec{0}\in\Sigma'_\iota[\lambda]$.

For the purpose, we shall evaluate the constant term of $\bar{S}_{-j_k}\cdots\bar{S}_{-j_1}(x_{-j;i})$ by using the similar method to the one in ([8], Lemma 5.2) and see that all of them are non-negative.

For $l\ge 0$, we define $\bar{S}^{(l)}_{-j;i}$ as follows:

$$\bar{S}^{(l)}_{-j;i}:=\bar{S}_{-j;i+l-1}\bar{S}_{-j;i+l-2}\cdots\bar{S}_{-j;i+1}\bar{S}_{-j;i}. \tag{5.5}$$

Now, we define

$$M^{(i)}:=\{\mu=(\mu_1,\mu_2,\cdots,\mu_i)\in\mathbf{Z}^i|n-i+1\ge\mu_1\ge\mu_2\ge\cdots\ge\mu_i\ge 0\},$$

whose element is called *i-admissible partition* ([8]). For $\mu\in M^{(i)}$ set

$$\varphi^{(\mu)}_{-j;i}(x):=\bar{S}^{(\mu_i)}_{-j+i-1;1}\bar{S}^{(\mu_{i-1})}_{-j+i-2;2}\cdots\bar{S}^{(\mu_{j+1})}_{1;i-j}\bar{S}^{(\mu_j)}_{-1;i-j+1}\bar{S}^{(\mu_{j-1})}_{-2;i-j+2}\cdots\bar{S}^{(\mu_2)}_{-j+1;i-1}\bar{S}^{(\mu_1)}_{-j;i}(x_{-j;i}) \tag{5.6}$$

By the similar arguement in [8] Lemma 5.2, we obtain the explicit form of $\varphi^{(\mu)}_{-j;i}(x)$ up to constant term:

Lemma 5.3. *We have*

$$\Xi'_\iota[\lambda]=\{\varphi^{(\mu)}_{-j;i}(x)+C_{-j;i}|j\ge 1,\ i\in I,\ \mu\in M^{(i)}\}, \tag{5.7}$$

$$\varphi^{(\mu)}_{-j;i}(x)-\varphi^{(\mu)}_{-j;i}(0)=\sum_{k=1}^{i}(x_{-j+k-\theta(j-k);i-k+1+\mu_k}-x_{-j+k+1-\theta(j-k-1);i-k+\mu_k}), \tag{5.8}$$

where we consider $x_{j:i}=0$ unless $1\le i\le n$ and

$$\theta(x):=\begin{cases}1 & \text{if } x\ge 0,\\ 0 & \text{if } x<0.\end{cases}$$

Let us evaluate the constant term in $\varphi^{(\mu)}_{-j;i}$. For $\mu=(\mu_1,\cdots,\mu_i)$, we set $\mu':=(\mu_1,\cdots,\mu_{j-1},0,\cdots,0)$ and $\mu'':=(\mu_1,\cdots,\mu_{j-1},\mu_j,0,\cdots,0)$ Then we know that $\varphi^{(\mu')}_{-j;i}$ has no constant term by its definition. Thus, be Lemma 5.3 we have

$$\begin{aligned} \varphi^{(\mu)}_{-j;i} &= (x_{-j;i+\mu_1}-x_{-j+1;i-1+\mu_1})+(x_{-j+1;i-1+\mu_2}-x_{-j+2;i-2+\mu_2})+\cdots \\ &\quad +(x_{-2;i-j+2+\mu_{j-1}}-x_{-1;i-j+1+\mu_{j-1}})+x_{-1;i-j+1}, \end{aligned}$$

By direct calculations, we obtain

$$\begin{aligned}
\varphi^{(\mu'')}_{-j;i} = \bar{S}^{(\mu_j)}_{-1;i-j+1}(\varphi^{(\mu')}_{-j;i}) &= \bar{S}_{-1;i-j+\mu_j}\cdots\bar{S}_{-1;i-j+1}(\varphi^{(\mu)}_{-j;i}) \\
&= \varphi^{(\mu)}_{-j;i} - x_{-1;i-j+\mu_j} + x_{-1;i-j+\mu_j+1} + \lambda_{i-j+1} + \lambda_{i-j+2} + \cdots + \lambda_{i-j+\mu_j}.
\end{aligned}$$

Since $\bar{S}^{(\mu_i)}_{-j+i-1;1}\bar{S}^{(\mu_{i-1})}_{-j+i-2;2}\cdots\bar{S}^{(\mu_{j+1})}_{1;i-j}$ does not produce non-trivial constant term, we have

$$\varphi^{(\mu'')}_{-j;i}(0) = \varphi^{(\mu)}_{-j;i}(0).$$

Hence, we obtain the constant term of $\varphi^{(\mu)}_{-j;i}(x)$,

$$\varphi^{(\mu)}_{-j;i}(0) = \lambda_{i-j+1} + \lambda_{i-j+2} + \cdots + \lambda_{i-j+\mu_j}.$$

By Lemma 5.1, we have

$$\begin{aligned}
&C_{-j;i} = (-\lambda_{-j+i+1} + (-\lambda_{-j+i+2} + (\cdots + (-\lambda_{-j+n+1})_+)_+)_+)_+ \\
&= \max(0, -\lambda_{-j+i+1}, -\lambda_{-j+i+1} - \lambda_{-j+i+2}, \cdots, -\lambda_{-j+i+1} - \cdots - \lambda_{n-j+1}),
\end{aligned}$$

and then

$$\begin{cases}
C_{-j;i} + \lambda_{i-j+1} \geq 0, \\
C_{-j;i} + \lambda_{i-j+1} + \lambda_{i-j+2} \geq 0, \\
\cdots, \\
C_{-j;i} + \lambda_{i-j+1} + \lambda_{i-j+2} + \cdots + \lambda_{n-j+1} \geq 0,
\end{cases} \tag{5.9}$$

which implies that $C_{-j;i} + \varphi^{(\mu)}_{-j;i}(0) \geq 0$. Therefore, constant terms in all elements in $\Xi'_\iota[\lambda]$ are non-negative and then we know that $\vec{0}$ is conatained in $\Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$. Therefore, we obtain $B_0(\lambda) \subset \Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$.

Finally, we will show that $\Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$ has the unique highest weight vector, which implies the opposite inclusion. We also show that the image of the highest weight vector by $\Psi^{(\lambda)}_\iota$ coincides with v_λ as in Theorem 5.2.

Let x_0 be the highest weight vector as in (5.1). We set

$$D_{-j;i} := \sigma_{-j;i}(x_0) + C_{-j;i}.$$

We shall show the following lemma on $D_{-j;i}$:

Lemma 5.4. *(i) If $D_{-j;i} \leq 0$, then we have $C_{-j;i} = 0$.*

(ii) If $D_{-j;i} > 0$, then we have

$$D_{-j;i} = (-\lambda_{-j+i+1} + (-\lambda_{-j+i+2} + (\cdots + (-\lambda_{-j+n+1})_+)_+)_+)_+ = C_{-j;i}.$$

Proof.

(I) The case $j = 1$.

By Lemma 5.1, we get

$$\begin{aligned}
D_{-1;i} &= (-\lambda_{i+1} + (-\lambda_{i+2} + (\cdots + (-\lambda_n)_+)_+\cdots)_+ - \lambda_i \\
&= \max(-\lambda_i, -\lambda_i - \lambda_{i+1}, \cdots, -\lambda_i - \lambda_{i+1} - \cdots - \lambda_n), \\
C_{-1;i} &= \max(0, -\lambda_i, -\lambda_i - \lambda_{i+1}, \cdots, -\lambda_i - \lambda_{i+1} - \cdots - \lambda_n) \\
&= \max(0, D_{-1;i})
\end{aligned}$$

If $D_{-1;i} \leq 0$, obviously $C_{-1;i} = 0$. If $D_{-1;i} > 0$, we obtain $D_{-1;i} = C_{-1;i}$ This shows that lemma is true.

(II) The case that $i = n$.
We will show by the induction on j. By the definition $D_{-j;i}$ and $C_{-j;i}$, we have

$$\begin{aligned} D_{-j;n} &= (-\lambda_{-j+n+1} + (-\lambda_{-j+n+2})_+)_+ - (-\lambda_{-j+n+2})_+ + \sigma_{-j+1;n}(x_0), \\ C_{-j;n} &= (-\lambda_{-j+n+1})_+. \end{aligned} \tag{5.10}$$

(i) The case $D_{-j+1;n} > 0$.
In this case, by the induction hypothesis, we have $\sigma_{-j+1;i}(x_0) = 0$. Then,

$$D_{-j;n} = (-\lambda_{-j+n+1} + (-\lambda_{-j+n+2})_+)_+ - (-\lambda_{-j+n+2})_+. \tag{5.11}$$

(i-a) The case $D_{-j;n} \leq 0$.
If $-\lambda_{-j+n+2} \leq 0$, then $-\lambda_{-j+n+1} \leq 0$. Thus, $C_{-j;n} = 0$. If $-\lambda_{-j+n+2} > 0$, then $-\lambda_{-j+n+1} \leq 0$ since $D_{-j;n} \leq 0$. Therefore, we obtain $C_{-j;n} = 0$.

(i-b) The case $D_{-j;n} > 0$.
Since the right hand-side of (5.11) is positive, we have

$$\max(0, -\lambda_{-j+n+1}, -\lambda_{-j+n+1} - \lambda_{-j+n+2}) > \max(0, -\lambda_{-j+n+2}). \tag{5.12}$$

This shows that the left hand-side of (5.12) is positive and equal to $-\lambda_{-j+n+1}$ or $-\lambda_{-j+n+1} - \lambda_{-j+n+2}$. If L.H.S of (5.12) $= -\lambda_{-j+n+1} > 0$, then $-\lambda_{-j+n+2} \leq 0$. Therefore, we obtain $D_{-j;n} = -\lambda_{-j+n+1} = C_{-j;n}$. If L.H.S of (5.12) $= -\lambda_{-j+n+1} - \lambda_{-j+n+2} > 0$, then $-\lambda_{-j+n+2} > 0$. Then $D_{-j;n} = -\lambda_{-j+n+1}$ must be positive. Thus, we obtain $C_{-j;n} = -\lambda_{-j+n+1} = D_{-j;n}$.

(ii) The case $D_{-j+1;n} \leq 0$.
By the induction hypothesis, we have $C_{-j+1;n} = (-\lambda_{-j+n+2})_+ = 0$ and then, by the definition of λ, $\lambda_{-j+n+1} \geq 0$. Then, by (5.11), we have

$$D_{-j;n} = \sigma_{-j+1;n}(x_0) \leq 0.$$

In this case, we have $C_{-j;n} = (-\lambda_{-j+n+1})_+ = 0$.

(III) The case $1 \leq i < j \leq n$.
By the definition, $C_{-j;i} = 0$. Then, we have

$$D_{-j;i} = \sigma_{-j;i}(x_0) \leq 0.$$

In this case, the lemma is true.

(IV) The case $1 < j \leq i < n$.
We have

$$D_{-j;i} = C_{-j;i+1} + C_{-j+1;i-1} - C_{-j+1;i} + \sigma_{-j+1;i}(x_0). \tag{5.13}$$

(i) The case $D_{-j+1;i} > 0$.

By the induction hypothesis, we have

$$D_{-j+1;n} = (-\lambda_{-j+i+2} + (-\lambda_{-j+i+3} + \cdots + (-\lambda_{-j+n+2})_+ \cdots)_+)_+ = C_{-j+1;i},$$

and then $\sigma_{-j+1;n}(x_0) = D_{-j+1;n} - C_{-j+1;n} = 0$. Then, we have

$$\begin{aligned}
D_{-j;i} &= (-\lambda_{-j+i+2} + (-\lambda_{-j+i+3} + \cdots + (-\lambda_{-j+n+1})_+) \cdots_+)_+ \\
&\quad + (-\lambda_{-j+i+1} + (-\lambda_{-j+i+2} + \cdots + (-\lambda_{-j+n+2})_+) \cdots_+)_+ \\
&\quad - (-\lambda_{-j+i+2} + (-\lambda_{-j+i+3} + \cdots + (-\lambda_{-j+n+2})_+) \cdots_+)_+ \\
&= \max(0, -\lambda_{-j+i+2}, -\lambda_{-j+i+2} - \lambda_{-j+i+3}, \cdots, -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+1}) \\
&\quad + (-\lambda_{-j+i+1} + \max(0, -\lambda_{-j+i+2}, -\lambda_{-j+i+2} - \lambda_{-j+i+3}, \cdots, -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+2}))_+ \\
&\quad - \max(\max(0, -\lambda_{-j+i+2}, -\lambda_{-j+i+2} - \lambda_{-j+i+3}, \cdots, -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+1}), \\
&\qquad\qquad -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+2})
\end{aligned}$$

Now, we set

$$\begin{aligned}
X &:= \max(0, -\lambda_{-j+i+2}, -\lambda_{-j+i+2} - \lambda_{-j+i+3}, \cdots, -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+1}) \geq 0, \\
Y &:= -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+2}.
\end{aligned}$$

Then,

$$D_{-j;i} = X + (-\lambda_{-j+i+1} + \max(X, Y))_+ - \max(X, Y)$$

(i-a) The case $\max(X, Y) = X$.
We have

$$\begin{aligned}
D_{-j;i} &= (-\lambda_{-j+i+1} + X)_+ \\
&= (-\lambda_{-j+i+1} + \max(0, -\lambda_{-j+i+2}, -\lambda_{-j+i+2} - \lambda_{-j+i+3}, \cdots, -\lambda_{-j+i+2} - \lambda_{-j+i+3} - \cdots - \lambda_{-j+n+1}))_+ \\
&= \max(0, -\lambda_{-j+i+1}, -\lambda_{-j+i+2} - \lambda_{-j+i+2}, \cdots, -\lambda_{-j+i+1} - \lambda_{-j+i+2} - \cdots - \lambda_{-j+n+1}) \\
&= C_{-j;i}.
\end{aligned}$$

In this case, the lemma is true.

(i-b) The case $\max(X, Y) = Y$.
In this case, we have

$$D_{-j;i} = X - Y + (-\lambda_{-j+i+1} + Y)_+.$$

First, we consider the case $X = 0$. If $D_{-j;i} > 0$, then $-\lambda_{-j+i+1} > 0$. By the definition of λ, we have $\lambda_{-j+i+1}, \lambda_{-j+i+2}, \cdots, \lambda_{-j+n+1} < 0$. This contradicts $X = 0$. Then, we consider the only case $D_{-j;i} \leq 0$. In this case, we have $-\lambda_{-j+i+1} \leq 0$. It follows from $X = 0$ that

$$\begin{cases}
-\lambda_{-j+i+1} \leq 0, \\
-\lambda_{-j+i+1} - \lambda_{-j+i+2} \leq 0, \\
\cdots \\
-\lambda_{-j+i+1} - \lambda_{-j+i+2} \cdots - \lambda_{-j+n+2} \leq 0.
\end{cases}$$

This shows $C_{-j;i}=0$. Therefore, since $X \geq 0$, we consider the case $X>0$.

There exist k such that

$$X \;=\; -\lambda_{-j+i+2}-\lambda_{-j+i+3}-\cdots-\lambda_{-j+k+1}>0.$$

Then we have

$$\begin{aligned}
X-Y &= \lambda_{-j+k+2}+\lambda_{-j+k+3}+\cdots+\lambda_{-j+n+2}<0,\\
D_{-j;i} &= X-Y+(-\lambda_{-j+i+1}+Y)_+\\
&= \max(X-Y,-\lambda_{-j+i+1}+X),\\
C_{-j;i} &= \max(0,-\lambda_{-j+i+1}+\max(-\lambda_{-j+i+2},-\lambda_{-j+i+2}-\lambda_{-j+i+3},\cdots-\lambda_{-j+i+2}-\cdots-\lambda_{-j+i+1}))\\
&= \max(0,-\lambda_{-j+i+1}+X).
\end{aligned}$$

If $D_{-j;i}\leq 0$, then $-\lambda_{-j+i+1}+X\leq 0$. This shows $C_{-j;i}=0$. If $D_{-j;i}>0$, since $X-Y<0$ we have $-\lambda_{-j+i+1}+X>0$, then $D_{-j;i}=-\lambda_{-j+i+1}+X=C_{-j;i}$. Therefore, if $D_{-j+1;i}>0$, we have shown the lemma.

(ii) The case $D_{-j+1;i}\leq 0$.
In this case, $C_{-j+1;i}=0$. Then, we have

$$\begin{cases}
-\lambda_{-j+i+2}\leq 0,\\
-\lambda_{-j+i+2}-\lambda_{-j+i+3}\leq 0,\\
\cdots\\
-\lambda_{-j+i+2}-\lambda_{-j+i+3}\cdots-\lambda_{-j+n+2}\leq 0.
\end{cases}$$

This shows, by the definition of λ, $\lambda_{-j+i+1}\geq 0$ and $-\lambda_{-j+i+1}+X\leq 0$. Since $0=C_{-j+1;i}\geq C_{-j;i+1}\geq 0$ we have $C_{-j;i+1}=0$. Then,

$$\begin{aligned}
D_{-j;i} &= C_{-j;i+1}+C_{-j+1;i-1}-C_{-j+1;i}+D_{-j+1;i}\\
&= (-\lambda_{-j+i+1}+\max(0,-\lambda_{-j+i+2},\cdots,-\lambda_{-j+i+2}-\lambda_{-j+i+3}-\cdots-\lambda_{-j+n+2}))_+ +D_{-j+1;i}\\
&= (-\lambda_{-j+i+1}+X)_+ +D_{-j+1;i}\\
&= D_{-j+1;i}\leq 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
C_{-j;i} &= \max(0,-\lambda_{-j+i+1}+\max(-\lambda_{-j+i+2},-\lambda_{-j+i+2}-\lambda_{-j+i+3},\cdots-\lambda_{-j+i+2}-\cdots-\lambda_{-j+i+1}))\\
&= \max(0,-\lambda_{-j+i+1}+X)=0.
\end{aligned}$$

This completes the proof of Lemma 5.4. $\square$

Let $v_0=(\cdots,0,0,t_\lambda,x_{-1;n},x_{-1;n-1},\cdots,x_{-j;i},\cdots)$ be a highest weight vector in $\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$ satisfying :

$$x_{-j;i}+C_{-j;i}\geq 0. \tag{5.14}$$

Note that the linear function $x_{-j;i}+C_{-j;i}$ in the left hand-side of the inequality is a generator of $\Xi'_\iota[\lambda]$. Thus, any vector in $\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$ satisfys the inequality (5.14).

Now, we shall show that v_0 is uniquely determined and coincides with v_0 by the induction on the index $(-j;i)$, where we consider the lexicographic order : $(-j;i)<(-j';i')$ if $j<j'$, or $j=j'$ and $i>i'$,*i.e.*,

$$(-1;n)<(-1;n-1)<\cdots<(-1;1)<(-2;n)<\cdots<\cdots(-j;i+1)<(-j;i)<(-j;i-1)<\cdots.$$

Recall the condition that x_0 is a highest weight vector :

$$\sigma_{-j;i}(v_0)\leq 0 \text{ for any}(-j;i)\ (j\geq 1, 1\leq i\leq n).$$

For $(-j;i)=(-n;1)$, we have $\sigma_{-1;n}(v_0)=x_{-1;n}-\lambda_n\leq 0$. By (5.14), we also have $x_{-1;n}+C_{-1;n}=x_{-1;n}-\lambda_n\geq 0$. Those imply

$$x_{-1;n}=\lambda_n=-C_{-1;n}.$$

Here note that by the assumption on λ, we have $\lambda_n\leq 0$. Asuume that for any $(-j';i')<(-j;i)$,

$$x_{-j';i'}=-C_{-j';i'}. \tag{5.15}$$

Let us determine $x_{-j;i}$. By this assumption (5.15), we have

$$\sigma_{-j;i}(v_0)=x_{-j;i}+D_{-j;i}. \tag{5.16}$$

If $D_{-j;i}\leq 0$, by Lemma 5.4(i) we have $C_{-j;i}=0$. Then we have

$$0\leq x_{-j;i}+C_{-j;i}=x_{-j;i}.$$

On the other hand, since v_0 is an element in $\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$, we have $x_{-j;i}\leq 0$. Those impliy $x_{-j;i}=0=-C_{-j;i}$.

If $D_{-j;i}>0$, by Lemma 5.4(ii) we have $D_{-j;i}=C_{-j;i}$. Since v_0 is a highest weight vector, we have

$$0\geq\sigma_{-j;i}(v_0)=x_{-j;i}+D_{-j;i}=x_{-j;i}+C_{-j;i}.$$

By the condition (5.14), we have

$$x_{-j;i}+C_{-j;i}\geq 0.$$

Thus, we obtain $x_{-j;i}=-C_{-j;i}$. Now, we know that v_0 is the unique highest weight vector in $\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$ satisfying $x_{-j;i}+C_{-j;i}\geq 0$. Since $B_0(\lambda)$ contains the unique highest weight vector ([4]), v_0 must be the unique highest weight vector in $B_0(\lambda)$, which implies that $B_0(\lambda)=\Sigma_\iota[\lambda]\cap\Sigma'_\iota[\lambda]$. □

6 Polyhedral Realization of $B(U_q(\mathfrak{g})a_\lambda)$ of Type $A_1^{(1)}$

In this section, we consider the case $\mathfrak{g}$ is of type $A_1^{(1)}$. We fix a positive level integral weight $\lambda=\lambda_1\Lambda_1+\lambda_2\Lambda_2$ ($\lambda_1\in\mathbf{Z}_{>0},\lambda_2\in\mathbf{Z}_{\leq 0}$, $\lambda_1+\lambda_2>0$). We define C_{-k} ($k\in\mathbf{Z}_{>0}$) as follows:

$$C_{-k}:=(-(k-1)\lambda_1-k\lambda_2)_+.$$

Theorem 6.1. *Let $\iota = (\cdots,2,1,2,1,t_\lambda,2,1,2,1,\cdots)$ be an infinite sequence and let $B_0(\lambda)$ be the connected component of* $\mathrm{Im}\,(\Psi_\iota^{(\lambda)})$ *containing* $\vec{0} := (\cdots,0,0,t_\lambda,0,0,\cdots)$. *Set*

$$\begin{aligned}
\Xi_\iota[\lambda] &:= \{\bar{S}_{j_l}\cdots\bar{S}_{j_1}(x_{j_0}) \,:\, l\geq 0,\, j_0,\cdots,j_l\neq 1\}\\
&\quad\cup\{\bar{S}_{-j_k}\cdots\bar{S}_{-j_1}(-x_{-j_0}) \,:\, k\geq 0,\, j_0,\cdots,j_k\neq 1\},\\
\Sigma_\iota[\lambda] &:= \{\vec{x}\in \mathbf{Z}_\iota^\infty[\lambda](\subset \mathbf{Q}^\infty) \,:\, \varphi(\vec{x})\geq 0 \text{ for any } \varphi\in\Xi_\iota[\lambda]\},\\
\Xi_\iota^{'}[\lambda] &:= \{\bar{S}_{-j_l}\cdots\bar{S}_{-j_1}(x_{-k}+C_{-k}) \,:\, l\geq 0,\, k\geq 1,\ j_1,\cdots,j_l\geq 1\},\\
\Sigma_\iota^{'}[\lambda] &:= \{\vec{x}\in \mathbf{Z}_\iota^\infty[\lambda](\subset \mathbf{Q}^\infty) \,:\, \varphi(\vec{x})\geq 0 \text{ for any } \varphi\in\Xi_\iota^{'}[\lambda]\}.
\end{aligned}$$

Then,

(i) $B_0(\lambda) = \Sigma_\iota[\lambda]\cap\Sigma'{}_\iota[\lambda]$.

(ii) Let v_λ be the highest weight vector of $B_0(\lambda)$. Then we have

$$v_\lambda = (\cdots,0,0,t_\lambda,-C_{-1},-C_{-2},\cdots,-C_{-k},\cdots).$$

Proof. Since $\Xi_\iota[\lambda]$ is closed by $\bar{S}_k$'s, by Lemma 4.3 it has a crystal structure. We will show that $\Sigma_\iota[\lambda]\cap\Sigma'{}_\iota[\lambda]$ contains $\vec{0}$ and has the unique highest weight vector. First, we will show that

$$x_0 = (\cdots,0,0,t_\lambda,-C_{-1},-C_{-2},\cdots,-C_{-k},\cdots)\ (1\leq k)$$

is one of the highest weight vectors of $\mathbf{Z}_\iota^\infty[\lambda]$. For $k\geq 1$, we recall the definition of σ_{-k}:

$$\sigma_{-k}(\vec{x}) = -\langle h_{i_{-k}},\lambda\rangle + x_{-k} + \sum_{j>-k}\langle h_{i_{-k}},\alpha_{i_j}\rangle x_j.$$

Let us show $\sigma_{-k}(x_0)\leq 0$ for $k\geq 1$ by the induction on k. (In the case $k<0$, trivially $\sigma_{-k}(x_0)=0$.)

(i) The case $k=1$.
we have $\sigma_{-1}(x_0) = -(-\lambda_2)_+ - \lambda_2 = 0$.

(ii) The case $k=2$.
By (5.2), we have

$$\begin{aligned}
\sigma_{-2}(x_0) &= -(-\lambda_1-2\lambda_2)_+ + 2(-\lambda_2)_+ - \lambda_1\\
&= -(-\lambda_1-2\lambda_2)_+ - 2\lambda_2 - \lambda_1\ (by\ \lambda_2\leq 0)\\
&\leq 0.
\end{aligned}$$

(iii) The case $k>2$.
We assume $\sigma_{-k+2}(x_0)\leq 0$. The following fact is trivial by $\lambda_1+\lambda_2>0$:

$$-(k-1)\lambda_1 - k\lambda_2 \leq 0 \implies -k\lambda_1-(k+1)\lambda_2\leq 0. \tag{6.1}$$

By the definition of $\sigma_{-k}(x_0)$, we heve

$$\sigma_{-k}(x_0) = -(-(k-1)\lambda_1-k\lambda_2)_+ + 2(-(k-2)\lambda_1-(k-1)\lambda_2)_+ - (-(k-3)\lambda_1-(k-2)\lambda_2)_+ + \sigma_{-k+2}(x_0).$$

Here, we set

$$X := -(k-1)\lambda_1 - k\lambda_2,\ Y := -(k-2)\lambda_1 - (k-1)\lambda_2,\ Z := -(k-3)\lambda_1 - (k-2)\lambda_2.$$

Then we have $\sigma_{-k}(x_0) = -(X)_+ + 2(Y)_+ - (Z)_+ + \sigma_{-k+2}(x_0)$.

By (6.1), if $Z \leq 0$, then $Y \leq 0$, $X \leq 0$ and if $Y \leq 0$, then $X \leq 0$. It is sufficient to show following four cases:

(iii-a) $Z \leq 0$. (iii-b) $Z > 0$, $Y \leq 0$. (iii-c) $Z > 0$, $Y > 0$, $X \leq 0$. (iii-d) $Z > 0$, $Y > 0$, $X > 0$.

(iii-a) The case $Z \leq 0$.
We have $\sigma_{-k}(x_0) = \sigma_{-k+2}(x_0) \leq 0$.

(iii-b) The case $Z > 0$, $Y \leq 0$.
By definition of $\sigma_{-k}(x_0)$, we have $\sigma_{-k}(x_0) = -Z + \sigma_{-k+2}(x_0) \leq 0$.

(iii-c) The case $Z > 0$, $Y > 0$, $X \leq 0$.

$$\begin{aligned}\sigma_{-k}(x_0) &= 2(-(k-2)\lambda_1 - (k-1)\lambda_2)_+ - (-(k-3)\lambda_1 - (k-2)\lambda_2)_+ + \sigma_{-k+2}(x_0)\\ &= X + \sigma_{-k+2}(x_0) \leq 0.\end{aligned}$$

(iii-d) The case $Z > 0$, $Y > 0$, $X > 0$.
$\sigma_{-k}(x_0) = -X + 2Y - Z + \sigma_{-k+2}(x_0) = \sigma_{-k+2}(x_0) \leq 0$.

Next, we will show that $\Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$ contains $\vec{0}$. By the similar way to the proof of Theorem5.2, we will show $\vec{0} \in \Sigma'_\iota[\lambda]$. For the purpose, we shall calculate the constant term of $\bar{S}_{-j_k} \cdots \bar{S}_{-j_1}(x_{-k})$.

We set

$$\varphi^{(l)}_{-k}(x) := \begin{cases} \bar{S}_{-k+l-1} \cdots \bar{S}_{-k+1}\bar{S}_{-k}(x_{-k}) & (l \leq k), \\ \bar{S}_{-k+l} \cdots \bar{S}_1\bar{S}_{-1}\bar{S}_{-2} \cdots \bar{S}_{-k+1}\bar{S}_{-k}(x_{-k}) & (l > k). \end{cases}$$

By the similar argument in [8] Lemma4.2, we obtain the explicit form of $\varphi^{(l)}_{-k}(x)$ up to constant term as follows:

Lemma 6.2.

$$\begin{aligned}&\Xi'_\iota[\lambda] = \{\varphi^{(l)}_{-k}(x) + C_{-k} \,|\, k \geq 1,\ l \geq 1\},\\ &\varphi^{(l)}_{-k}(x) - \varphi^{(l)}_{-k}(0) = (l+1)x_{l-k+\theta(l-k)} - l x_{l-k+1+\theta(l-k+1)}\end{aligned}$$

where

$$\theta(x) := \begin{cases} 1 & \textit{if } x \geq 0, \\ 0 & \textit{if } x < 0. \end{cases}$$

Now, we calculate the constant term in $\varphi^{(l)}_{-k}$. For $l \leq k-2$, we know that $\varphi^{(l)}_{-k}$ has no costant term by its definition. And we have

$$\varphi^{(k-2)}_{-k}(x) = (k-1)x_{-2} - (k-2)x_{-1}.$$

By direct calculations, we obtain

$$\begin{aligned}
\varphi_{-k}^{(k-1)} &= \bar{S}_{-2}(\varphi_{-k}^{(k-2)}) \\
&= kx_{-1} - (k-1)x_1 + (k-1)\lambda_1, \\
\varphi_{-k}^{(k)} &= \bar{S}_{-1}\bar{S}_{-2}(\varphi_{-k}^{(k-2)}) \\
&= (k+1)x_1 - kx_2 + (k-1)\lambda_1 + k\lambda_2.
\end{aligned}$$

For $k \geq 1$, since $\bar{S}_k$ does not produce non-trivial constant term, we have

$$\varphi_{-k}^{(k)}(0) = \varphi_{-k}^{(l)}(0) \ (l > k).$$

Hence, we obtain the constant term of $\varphi_{-k}^{(l)}(x)$:

$$\varphi_{-k}^{(l)}(0) = \begin{cases} (k-1)\lambda_1 + k\lambda_2 & (l \geq k), \\ (k-1)\lambda_1 & (l = k-1), \\ 0 & (l \leq k-2). \end{cases} \tag{6.2}$$

By Lemma 5.1, we have $C_{-k} + \varphi_{-k}^{(l)}(0) \geq 0$. This shows that constant terms in all elements in $\Xi_\iota'[\lambda]$ are non-negative and then $\vec{0}$ is contained in $\Sigma_\iota[\lambda] \cap \Sigma_\iota'[\lambda]$. Therefore, we have $B_0(\lambda) \subset \Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$.

Finally, we will show that $\Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$ has the unique highest weight vector v_λ.

We define

$$D_{-k} := \sigma_{-k}(x_0) + C_{-k}.$$

Note that for $k \geq 3$,

$$\begin{aligned}
D_{-k} &= 2(-(k-2)\lambda_1 - (k-1)\lambda_2)_+ - (-(k-3)\lambda_1 - (k-2)\lambda_2)_+ + \sigma_{-k+2}(x_0) \\
&= 2(Y)_+ - (Z)_+ + \sigma_{-k+2}(x_0), \qquad (6.3) \\
C_{-k} &= (-(k-1)\lambda_1 - k\lambda_2)_+ \\
&= (X)_+. \qquad (6.4)
\end{aligned}$$

We need the following lemma:

Lemma 6.3. *(i) If $D_{-k} \leq 0$, then $C_{-k} = 0$.*

(ii) If $D_{-k} > 0$, then we have $D_{-k} = (-(k-1)\lambda_1 - k\lambda_2)_+ = C_{-k}$.

Proof. We shall show the lemma by the induction on k.

(I) The case $k = 1$.

By the definition of D_{-1}, we have $D_{-1} = -\lambda_2 > 0$. This shows $C_{-1} = (-\lambda_2)_+ = -\lambda_2$.

(II) The case $k = 2$.

By the definition of D_{-2} and C_{-2}, we have

$$D_{-2} = 2(-\lambda_2)_+ - \lambda_1 = -2\lambda_2 - \lambda_1, \quad C_{-2} = (-\lambda_1 - 2\lambda_2)_+ = (D_{-2})_+.$$

Therefore, we get if $D_{-2} \leq 0$, then $C_{-2} = 0$. If $D_{-2} > 0$, then $D_{-2} = -\lambda_1 - 2\lambda_2 = C_{-2}$.

(III) The case $k \geq 3$.

(i) The case $D_{-k+2} > 0$.
By the induction hypothesis, $\sigma_{-k+2}(x_0) = D_{-k+2} - C_{-k+2} = 0$. By (6.3) and (6.4)

$$D_{-k} = 2(Y)_+ - (Z)_+, \qquad C_{-k} = (X)_+.$$

By (6.1), if $Z \leq 0$, then $Y \leq 0$ and $X \leq 0$. In this case, we have $D_{-k} = 0$ and $C_{-k} = 0$. If $Y \leq 0$, then $X \leq 0$. In this case, $D_{-k} = -(Z)_+ \leq 0$ and $C_{-k} = 0$. Then, it is sufficient to show following two cases:

(i-a) $Z > 0$, $Y > 0$ and $X \leq 0$.

(i-b) $Z > 0$, $Y > 0$ and $X > 0$.

(i-a) The case $Z > 0$, $Y > 0$ and $X \leq 0$.
We have

$$\begin{aligned} D_{-k} &= 2(-(k-2)\lambda_1 - (k-1)\lambda_2) - (-(k-3)\lambda_1 - (k-2)\lambda_2) \\ &= -(k-1)\lambda_1 - k\lambda_2 = X \leq 0, \\ C_{-k} &= (X)_+ = 0. \end{aligned}$$

This show that the lemma is true.
(i-b) The case $Z > 0$, $Y > 0$ and $X > 0$.
We have

$$\begin{aligned} D_{-k} &= 2(-(k-2)\lambda_1 - (k-1)\lambda_2) - (-(k-3)\lambda_1 - (k-2)\lambda_2) \\ &= -(k-1)\lambda_1 - k\lambda_2 = X > 0, \\ C_{-k} &= (X)_+ = X = D_{-k}. \end{aligned}$$

Therefore, in the case $D_{-k+2} > 0$, the lemma is true.
(ii) The case $D_{-k+2} \leq 0$.
By the induction hypothesis, $C_{-k+2} = (-(k-3)\lambda_1 - (k-2)\lambda_2)_+ = 0$, and then $Z = -(k-3)\lambda_1 - (k-2)\lambda_2 \leq 0$. By (6.1), Y, $X \leq 0$. By (refdefd) and (6.4), we have

$$D_{-k} = \sigma_{-k+2}(x_0) \leq 0, \qquad C_{-k} = 0.$$

Therefore, we complete the proof. $\square$

Let $v_0 := (\cdots, 0, 0, t_\lambda, x_{-1}, x_{-2}, \cdots, x_{-k}, \cdots)$ be a highest weight vector in $\Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$, which satisfys:

$$x_{-k} + C_{-k} \geq 0. \tag{6.5}$$

Note that the linear function $x_{-k} + C_{-k}$ is a generator of $\Xi'_\iota[\lambda]$, and then any vector in $\Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$ satisfys the inequality (6.5).

We shall show that v_0 is uniquely determined and coincides with v_λ by the induction on the index k.

By the condition that v_0 is a highest weight vector, we have that $\sigma_{-k}(v_0) \leq 0$ for any $k \geq 1$. For $k = 1$, we have $\sigma_{-1}(v_0) = x_{-1} - \lambda_2 \leq 0$. By (6.5), we also have $x_{-1} + C_{-1} = x_{-1} - \lambda_2 \geq 0$, which implies $x_{-1} = \lambda_2 = -C_{-1}$. Assume that for any $k' < k$,

$$x_{-k'} = -C_{-k'}. \tag{6.6}$$

Let us determine x_{-k}. By the assumption (6.6), we have

$$\sigma_{-k}(v_0) = x_{-k} + D_{-k}. \tag{6.7}$$

If $D_{-k} \leq 0$, by Lemma 6.3(i) we have $C_{-k} = 0$. Then we have $0 \leq x_{-k} + C_{-k} = x_{-k}$. On the other hand, since v_0 is an element in $\Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$, we have $x_{-k} \leq 0$. We obtain $x_{-k} = 0 = -C_{-k}$.

If $D_{-k} > 0$, by Lemma 6.3(ii) we have $D_{-k} = C_{-k}$. Since v_0 is a highest weight vector, we have

$$0 \geq \sigma_{-k}(v_0) = x_{-k} + D_{-k} = x_{-k} + C_{-k}.$$

On the other hand, by the condition (6.5), we have $x_{-k} + C_{-k} \geq 0$. Thus, we obtain $x_{-k} = -C_{-k}$. Now, we know that v_0 is the unique highest weight vector in $\Sigma_\iota[\lambda] \cap \Sigma'_\iota[\lambda]$ satisfying $x_{-k} + C_{-k} \geq 0$. Since $B_0(\lambda)$ contains the unique highest weight vector ([4]), v_0 must be the unique highest weight vector in $B_0(\lambda)$. Hence, we have $B_0(\lambda) = \Sigma_\iota[\lambda] \cap {\Sigma'}_\iota[\lambda]$. □

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